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Electron Scattering without Spin Sums

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Using the spacetime algebra formulation of the Dirac equation, we demonstrate how to perform cross-section calculations following a method suggested by Hestenes (1982). Instead of an *S*-matrix, we use an operator that rotates the initial states into the scattered states. By allowing the scattering operator to become a function of the initial spin, we can neatly handle spin-dependent calculations. When the operator is independent of spin, we can provide manifestly spin-independent results. We use neither spin basis nor spin sums, instead handling the spin orientation directly. As examples, we perform spin-dependent calculations in Coulomb scattering to second order, and briefly consider more complicated calculations in QED.

1. INTRODUCTION

Calculation of spinor scattering cross sections usually involves complicated abstract calculations with gamma matrices. In this paper, we perform cross-section calculations in a more transparent and intuitive way. We incorporate the spin orientation directly rather than summing over spins and using spin projection operators. This streamlines the calculation of spin-dependent results and makes it clear when results are independent of spin. Our method is most easily applied to single-electron scattering, which we discuss first, but we follow with a brief discussion of how to handle multiparticle scattering.

The starting point for our approach is the spacetime algebra formulation of Dirac theory. The spacetime algebra (STA) is the geometric (Clifford) algebra of Minkowski spacetime, first developed by Hestenes (1966, 1975, 1982). The formulation of Dirac theory within the algebra replaces the matrices of the conventional theory with multivectors. The two formulations are entirely equivalent, but the STA approach brings out the geometric structure

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leading to more physically transparent calculations. We briefly summarize the STA formulation below and then explain a method for performing cross-section calculations first demonstrated by Hestenes (1982). We extend and clarify this work, handling spin dependence in a natural way.

2. SPACETIME ALGEBRA AND THE DIRAC EQUATION

Throughout we shall make use of the geometric algebra. We present a brief summary of the STA below to clarify our notation and conventions (Hestenes, 1966; Hestenes and Sobczyk, 1984).

We shall use the four orthogonal basis vectors of spacetime γ_μ, where $\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ for $k = 1, 2, 3$. The geometric algebra has an associative product, and the basis vectors then satisfy the Dirac algebra

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \text{diag}(+---)$$

The antisymmetric part of the product defines the *outer product*

$$\gamma_\mu \wedge \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

By repeated multiplication of the basis vectors, we can build up the 16 basis elements of STA multivectors:

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{I\gamma_\mu\}$	I
scalar	vectors	bivectors	pseudovectors	pseudoscalar

We can do a space-time split of a vector into the γ_0 frame by multiplying by γ_0. For example, the momentum p is split as follows:

$$p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0 = E + \mathbf{p} = p^0 + p^i\gamma_i\gamma_0$$

Bold letters are now used for relative 3-vectors (spacetime bivectors). Restricted Lorentz transformations are spacetime rotations and can be performed by use of a *rotor*, which can be written $R = \pm \exp(B/2)$. Here B is a bivector in the plane of the rotation and $|B|$ determines the amount of rotation. The rotation of a multivector M is then given by

$$M \rightarrow RM\tilde{R}$$

In the STA, spinors are represented using the even subalgebra, which has the required eight degrees of freedom. The minimally coupled form of the Hestenes' Dirac equation is

$$\hat{j}\nabla\psi\gamma_0 - m\psi = eA\psi\gamma_0$$

Here $\hat{j}$ is an operator that multiplies on the right by an arbitary reference spatial bivector $\boldsymbol{\Sigma}$ so that

$$\hat{j}\psi \equiv \psi\mathbf{\Sigma}$$

The spinor ψ can be decomposed as

$$\psi = \rho^{1/2}\, e^{I\beta/2}\, LU$$

where L and U are rotors for a boost and spatial rotation respectively, ρ is a scalar, and the β-factor determines the rest ratio of particles and antiparticles. So in the STA approach, the spinor directly encodes a Lorentz transformation and a propability density. The rotor U rotates the arbitrary reference plane $\mathbf{\Sigma}$ into the *rest spin bivector* observable of the electron

$$\hat{S}^0 \equiv U\mathbf{\Sigma}\tilde{U}$$

and the boost L gives the momentum

$$p = mL\gamma_0\tilde{L}$$

We can also boost the rest spin bivector to define the relativistic spin bivector

$$\hat{S} = L\hat{S}^0\tilde{L} = \rho^{-1}\psi\mathbf{\Sigma}\tilde{\psi}$$

Positive- and negative-energy plane wave solutions are given as usual by

$$\psi^{(+)} = u(p)e^{-\hat{j}p\cdot x} \qquad \text{and} \qquad \psi^{(-)} = v(p)e^{\hat{j}p\cdot x}$$

and the energy projection operators are

$$\Lambda_{\pm}(\psi) = \frac{1}{2m}\,(m\psi \pm p\psi\gamma_0)$$

3. THE FEYNMAN PROPAGATOR

The Feynman propagator S_F is the Greens' function for the Dirac equation that propagates negative-energy waves into the past and positive-energy waves into the future. As a Greens' function, it satisfies

$$\hat{j}\nabla_x S_F(x - x')\psi(x')\gamma_0 - mS_F(x - x')\psi(x') = \delta^4(x - x')\psi(x')$$

and an integral solution to the Dirac equation is given by

$$\psi(x) = \psi_i(x) + e\int d^4x'\; S_F(x - x')A(x')\psi(x')\gamma_0 \tag{1}$$

where ψ_i satisfies the free-particle equation. Taking the Fourier transform, we have

$$pS_F(p)\psi\gamma_0 - mS_F(p)\psi = \psi$$

$$S_F(x - x') = \int \frac{d^4p}{(2\pi)^4}\, S_F(p)e^{-\hat{j}p\cdot(x-x')}$$

Operating on both sides with the energy projection operator Λ_+, we can solve for the momentum-space Feynman propagator:

$$(p^2 - m^2)S_F(p)\psi = p\psi\gamma_0 + m\psi$$
$$\Rightarrow S_F(p)\psi = \frac{p\psi\gamma_0 + m\psi}{p^2 - m^2 + \hat{j}\epsilon} \tag{2}$$

The $\hat{j}\epsilon$ ensures causality: positive-energy waves propagate into the future and negative-energy waves into the past. Fourier-transforming back and performing the integral over dE, we have

$$S_F(x - x')\psi = -2m\hat{j}\int \frac{d^3\mathbf{p}}{2E_p(2\pi)^3}\left[\theta(t - t')\Lambda_+(\psi)e^{-\hat{j}p\cdot(x-x')} + \theta(t' - t)\Lambda_-(\psi)e^{\hat{j}p\cdot(x-x')}\right] \tag{3}$$

where $E = +\sqrt{\mathbf{p}^2 + m^2}$.

4. ELECTRON SCATTERING

For scattering calculation, we write the wavefunction as the sum of an incoming plane wave and a scattered wave, $\psi = \psi_i + \psi_{\text{diff}}$, where ψ_{diff} is the solution at late times given by

$$\psi_{\text{diff}}(x) = -2m\hat{j}e\int d^4x' \int \frac{d^3\mathbf{p}}{2E_p(2\pi)^3}\,\Lambda_+[A(x')\psi(x')\gamma_0]e^{-\hat{j}p\cdot(x-x')}$$

We can write this as a sum over final states

$$\psi_{\text{diff}}(x) = \int \frac{d^3\mathbf{p}_f}{2E_f(2\pi)^3}\,\psi_f(x)$$

where the final states are plane waves,

$$\psi_f(x) \equiv \psi_f e^{-\hat{j}p_f x} \equiv -\hat{j}e\int d^4x'\,[p_f A(x')\psi(x') + mA(x')\psi(x')\gamma_0]e^{-\hat{j}p_f\cdot(x-x')} \tag{4}$$

To generate the Born series perturbative solution, we iterate (1). The first-order Born approximation amounts to simply replacing $\psi(x')$ by $\psi_i(x')$. For plane waves of particles, we have

$$\psi(x) = \psi e^{-\hat{j}p\cdot x} \qquad \text{and} \qquad m\psi\gamma_0 = p\psi$$

$$\psi_f = -\hat{j}e \int d^4x' \, [p_f A(x') + A(x')p_i] \, \psi_i e^{\hat{j}q\cdot x'}$$

$$= -\hat{j}e[p_f A(q) + A(q)p_i]\psi_i, \qquad q \equiv p_f - p_i$$

In general, we define a *scattering operator* S_{fi} by

$$\psi_f = S_{fi}\psi_i$$

This rotates and dilates the initial states into the final states. The f and i indices label the initial and final momenta and the initial spin, so in general $S_{fi} = S_{fi}(p_f, p_i, \hat{S}_i)$. However S_{fi} does not depend on the final spin, instead the final spin is determined from the initial spin by a rotation encoded in S_{fi}.

Since S_{fi} consists of a rotation and dilation, it is convenient to decompose it as

$$S_{fi} = \rho_{fi}^{1/2} R_{fi}$$

where R_{fi} is a rotor. The ρ_{fi} factor determines the cross section, as discussed in the next section. The rotor R_{fi} rotates states with momentum p_i into states with momentum p_f. It also rotates the initial spin into the final spins by

$$\hat{S}_f = R_{fi}\hat{S}_i\tilde{R}_{fi}$$

The rest spins are therefore related by

$$\hat{S}_f^0 = \tilde{L}_f\hat{S}_f L_f = \tilde{L}_f R_{fi}\hat{S}_i\tilde{R}_{fi}L_f = \tilde{L}_f R_{fi}L_i\hat{S}_i^0\tilde{L}_i\tilde{R}_{fi}L_f$$

We define the *rest spin scattering operator*

$$U_{fi} \equiv \tilde{L}_f R_{fi} L_i, \qquad \hat{S}_f^0 = U_{fi}\hat{S}_i^0\tilde{U}_{fi}$$

The cross section and rest spin scattering operator contain all the information about scattering of states with initial momentum p_i and spin $\hat{S}_i$ into final states with momentum p_f.

The external line Feynman propagator is given by (3) and ensures that S_{fi} is of the form

$$S_{fi} = -\hat{j}(p_f M + M p_i) \tag{5}$$

where in the Born approximation example, $M = eA(q)$. However, in general M can depend on $\hat{j}$, in which case we can write

$$S_{fi}\psi_i = -\hat{j}(p_f[M_r + \hat{j}M_j] + [M_r + \hat{j}M_j]p_i)\psi_i$$

where M_j and M_r are independent of $\hat{j}$. Using $\hat{j}\psi_i = \psi_i\mathbf{\Sigma} = \hat{S}_i\psi_i$ and the fact that $\hat{S}_i$ and p_i commute, we can write this as

$$S_{fi} = -\hat{j}(p_f M + M p_i)$$

where now $M = M_r + M_j\hat{S}_i$ depends on the initial spin. We can therefore convert dependence on the 'imaginary' $\hat{j}$ into dependence on the physical spin bivector.

5. CROSS SECTIONS

The scattering rate into the final states per unit volume per unit time is given by

$$W_{fi} = \frac{\rho_f}{2mVT}, \qquad \rho_f = |S_{fi}|^2\rho_i = \rho_{fi}\rho_i$$

The cross section is defined as

$$d\sigma = \frac{W_{fi}}{\text{Target density} \times \text{Incident flux}}$$

For elastic scattering, we have

$$S_{fi} = -\hat{j}2\pi\delta(E_f - E_i)T_{fi}, \qquad |S_{fi}|^2 = 2\pi T\delta(E_f - E_i)|T_{fi}|^2$$

With a target density of $1/V$ and an incident flux of $|\mathbf{J}_i| = \rho_i|\mathbf{p}_i|/m$, we have

$$d\sigma = \frac{\pi}{|\mathbf{p}_i|}\,\delta(E_f - E_i)|T_{fi}|^2$$

This is readily extended to positron scattering and to more complicated cases.

6. COULUMB SCATTERING

Coulomb scattering is a useful test case where the vector potential is given by

$$A(x) = \frac{-Ze}{4\pi|\mathbf{x}|}\,\gamma_0$$

Taking the Fourier transform, we have

$$A(q) = -\frac{2\pi Ze}{\mathbf{q}^2}\,\delta(E_f - E_i)\gamma_0$$

and $M = eA(q)$ in the first Born approximation. Writing

$$S_{fi} = -\hat{j}2\pi\delta(E_f - E_i)T_{fi}$$

and using energy conservation, we have

$$T_{fi} = -\frac{Ze^2}{\mathbf{q}^2}(2E + \mathbf{q})$$

so that the formula for the cross section becomes

$$d\sigma = \left(\frac{Ze^2}{\mathbf{q}^2}\right)^2 \frac{\pi}{|\mathbf{p}_i|}\,\delta(E_f - E_i)(4E^2 - \mathbf{q}^2)\,\frac{d^3\mathbf{p}_f}{2E_f(2\pi)^3}$$

Using $d^3\mathbf{p}_f = |\mathbf{p}_f|E_f\,dE_f d\Omega_f$, we recover the Mott cross section

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Mott}} = \frac{Z^2\alpha^2}{\mathbf{q}^4}(4E^2 - \mathbf{q}^2) = \frac{Z^2\alpha^2}{4\mathbf{p}^2\beta^2\sin^4(\theta/2)}[1 - \beta^2\sin^2(\theta/2)]$$

where $\mathbf{q}^2 = (\mathbf{p}_f - \mathbf{p}_i)^2 = 2\mathbf{p}^2(1 - \cos\theta)$ and $\beta = |\mathbf{p}|/E$. The derivation is manifestly independent of initial spin, and the cross section is therefore spin independent. If we had instead used the conventional spin sums method, this would have been far from clear.

The final and initial spins will be related by the rest spin scattering operator U_{fi}, where

$$U_{fi} \propto L_f L_i + \tilde{L}_f\tilde{L}_i \propto (E + m)^2 + \mathbf{p}_f\mathbf{p}_i$$

If U_{fi} rotates by an angle δ in the $\hat{B}$ plane ($\hat{B}^2 = -1$), it is given by

$$U_{fi} = e^{\delta\hat{B}/2} = \cos(\delta/2) + \hat{B}\sin(\delta/2)$$

The rotation is therefore in the $\mathbf{p}_f \wedge \mathbf{p}_i$ plane and by an angle δ given by

$$\tan\delta/2 = \frac{|\langle U_{fi}\rangle_2|}{\langle U_{fi}\rangle} = \frac{|\mathbf{p}_f \wedge \mathbf{p}_i|}{(E + m)^2 + \mathbf{p}_f \cdot \mathbf{p}_i} = \frac{\sin\theta}{(E + m)/(E - m) + \cos\theta}$$

7. SECOND-ORDER COULUMB SCATTERING

Second-order Coulomb scattering is interesting, as it is spin-dependent, though the calculation is now rather more involved. To avoid problems with divergent integrals, we replace the potential with the screened potential

$$A(x) = -\frac{e^{-\lambda|\mathbf{x}|}Ze}{4\pi|\mathbf{x}|}\gamma_0$$

and obtain the Coulomb result in the limit as λ goes to zero (Dalitz, 1951;

Itzykson and Zuber, 1980). For this potential, the first-order analysis above can be applied with M given by

$$eA(q) = -\frac{2\pi Ze^2}{\lambda^2 + \mathbf{q}^2}\,\delta(E_f - E_i)\gamma_0$$

To iterate to second order, (4) is used, substituting

$$\psi(x') = \psi_i\, e^{-\hat{\jmath}p_i\cdot x'} + e\int d^4x''\int \frac{d^4k}{(2\pi)^4}\,\frac{kA(x'') + A(x'')p_i}{k^2 - m^2 + \hat{\jmath}\epsilon}\,\psi_i\, e^{\hat{\jmath}x''\cdot(k-p_i)}\, e^{-\hat{\jmath}k\cdot x'}$$

giving the extra contribution to M

$$M' = e^2\int d^4x'\int d^4x''\int \frac{d^4k}{(2\pi)^4}\,A(x')\,\frac{kA(x'') + A(x'')p_i}{k^2 - m^2 + \hat{\jmath}\epsilon}\, e^{\hat{\jmath}x'\cdot(p_f-k)}\, e^{\hat{\jmath}x''\cdot(k-p_i)}$$

Performing the x' and x'' integrations and using one of the resultant δ-functions, we have

$$M = 2\pi\delta(E_f - E_i)M_T$$

where the extra contribution to M_T is

$$M_T' = e^2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{a_0(p_f - k)a_0(k - p_i)}{k^2 - m^2 + \hat{\jmath}\epsilon}\,\gamma_0[k\gamma_0 + \gamma_0 p_i]$$

$$a_0(p) = \int d^3\mathbf{x}\, e^{-p\cdot x}\gamma_0\cdot A(\mathbf{x}) = \frac{-Ze}{\lambda^2 + \mathbf{p}^2}$$

Using

$$k^2 - m^2 = \mathbf{p}_i^2 - \mathbf{k}^2$$

and the integrals

$$I_1 + \tfrac{1}{2}(\mathbf{p}_i + \mathbf{p}_f)I_2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1 + \mathbf{k}}{[(\mathbf{p}_f - \mathbf{k})^2 + \lambda^2][(\mathbf{p}_i - \mathbf{k})^2 + \lambda^2](\mathbf{p}_i^2 - \mathbf{k}^2 + \hat{\jmath}\epsilon)}$$

we have

$$M_T' = Z^2\, e^4[\gamma_0\tfrac{1}{2}(\mathbf{p}_i + \mathbf{p}_f)I_2 + (p_i + \gamma_0 E)I_1]$$

We take the limit $\lambda \to 0$, and the total M_T to second order is then

$$M_T = \frac{-Ze^2}{\mathbf{q}^2}\,\gamma_0 + Z^2\, e^4[(E\gamma_0 - \tfrac{1}{2}[p_f + p_i])I_2 + (p_i + \gamma_0 E)I_1]$$

where the integrals are (Itzykson and Zuber, 1980)

$$I_1 = \frac{-\hat{j}}{16\pi|\mathbf{p}|^3 \sin^2(\theta/2)} \ln \frac{2|\mathbf{p}|\sin(\theta/2)}{\lambda}$$

$$I_2 = \frac{1}{16\pi|\mathbf{p}|^3 \cos^2(\theta/2)} \left\{ \frac{\pi[\sin(\theta/2) - 1]}{2 \sin^2(\theta/2)} - \hat{j} \ln \frac{\lambda}{2|\mathbf{p}|} \right\} + \frac{I_1}{\cos^2(\theta/2)}$$

This time M has some $\hat{j}$ dependence and we write $I_1 = (A + C)\hat{j}$ and $I_2 = B + C\hat{j}$, where A, B, and C are scalars. Replacing the $\hat{j}$ dependence with $\hat{S}_i$ dependence, we have

$$M_T = \gamma_0 \left[-\frac{Ze^2}{\mathbf{q}^2} + EZ^2 e^4 \{B + (2C + A)\hat{S}_i\} \right]$$

$$+ Z^2 e^4 \left[p_i(A\hat{S}_i - B) - \frac{1}{2} q(B + C\hat{S}_i) \right]$$

The term proportional to q cancels in the calculation of T_{fi}. Using the result that

$$p_f p_i + m^2 = E(2E + \mathbf{q}) - \mathbf{p}^2 - \mathbf{p}_f \mathbf{p}_i$$

we have

$$T_{fi} = (2E + \mathbf{q}) \left[-\frac{Ze^2}{\mathbf{q}^2} + 2EZ^2 e^4 (A + C)\hat{S}_i \right]$$

$$+ Z^2 e^4 (\mathbf{p}^2 + \mathbf{p}_f \mathbf{p}_i)(B - A\hat{S}_i)$$

Keeping terms up to α^3, the cross section is governed by

$$|T_{fi}|^2 = (4E^2 - \mathbf{q}^2) \frac{Z^2 e^4}{\mathbf{q}^4} - \frac{4Z^3 e^6}{\mathbf{q}^2} [EB(\mathbf{p}^2 + \mathbf{p}_f \cdot \mathbf{p}_i) + mA(\mathbf{p}_i \wedge \mathbf{p}_f) \cdot \hat{S}_i^0]$$

where $\hat{S}_i^0$ is the initial rest spin. The divergent parts of the integrals have canceled out and we are only left with the finite terms B and

$$A = \frac{\ln \sin(\theta/2)}{16\pi|\mathbf{p}|^3 \cos^2(\theta/2)}$$

We could obtain the cross section for unpolarized scattering by averaging over the initial spin. The result is the spin-independent part of the cross section since the spin-dependent part averages to zero.

8. SPIN DEPENDENCE AND DOUBLE SCATTERING

We now calculate the asymmetry parameter for double scattering from a Coulomb potential as an example of a spin-dependent calculation. The

idea is that since the second-order correction to Coulomb scattering is spin dependent, the scattered beam will be partially polarized even with an unpolarized incident beam. The scattered beam can then impinge on a second target, giving an observable asymmetry in the scattered intensity. This asymmetry was first worked out by Mott (1929, 1932).

The spin after the first scattering is given by

$$\hat{S}_f = R_{fi}\hat{S}_i\tilde{R}_{fi}$$

$$\hat{S}_f \propto T_{fi}\hat{S}_i\tilde{T}_{fi} = \frac{Z^2\,e^4}{\mathbf{q}^4}(2E + \mathbf{q})\hat{S}_i(2E - \mathbf{q}) - \frac{2Z^3\,e^6 A}{\mathbf{q}^2}\langle(\mathbf{p}^2 + \mathbf{p}_f\mathbf{p}_i)(2E - \mathbf{q})\rangle_2$$

where we have only kept the lowest order terms in the spin-dependent and spin-independent parts. The first term depends on the initial spin, but the second term does not, so if we average over the initial spin, the spin-independent part will determine the final polarization. We define S^0 to be the polarization in the plane $\hat{S}^0$. This is a bivector in the plane of $\hat{S}^0$ with modulus equal to the polarization of the beam. Since the incoming beam is unpolarized, the resultant polarization plane will be given by the spin-independent part of $\hat{S}_f$ deboosted to rest. To get the polarization, we then divide by the magnitude of the spin-dependent part, obtaining

$$S_f^0 = -\frac{2Z\,e^2\mathbf{q}^2 A}{(4E^2 - \mathbf{q}^2)}\tilde{L}_f\langle(\mathbf{p}^2 + \mathbf{p}_f\mathbf{p}_i)(2E - \mathbf{q})\rangle_2 L_f$$

$$= \frac{2Z\,e^2\mathbf{q}^2 A}{(4E^2 - \mathbf{q}^2)}\,2m\mathbf{p}_i \wedge \mathbf{p}_f$$

The beam after the first scattering is therefore polarized in the scattering plane $\mathbf{p}_i \wedge \mathbf{p}_f$. The spin-dependent part of the cross section for the second scattering is given by

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{spin}} = -\frac{4Z^3\,e^6 m A_2}{\mathbf{q}_2^2(2\pi)^2}(\mathbf{p}_f \wedge \mathbf{p}_2)\cdot S_f^0$$

$$= -\frac{64(2\pi)^2 Z^4\alpha^4\mathbf{q}_1^2 m^2 A_1 A_2}{\mathbf{q}_2^2(4E^2 - \mathbf{q}_1^2)}(\mathbf{p}_f \wedge \mathbf{p}_2)\cdot(\mathbf{p}_i \wedge \mathbf{p}_f)$$

where the subscripts 1 and 2 refer to the first are second scattering, respectively (e.g., $\mathbf{q}_2 = \mathbf{p}_2 - \mathbf{p}_f$). The asymmetry therefore depends on the cosine of the angle ϕ between the $\mathbf{p}_f \wedge \mathbf{p}_2$ and $\mathbf{p}_i \wedge \mathbf{p}_f$ planes. The asymmetry parameter δ is defined so that the final intensity depends on ϕ through the factor

$$1 + \delta \cos \phi$$

In the case where $\mathbf{p}_i \cdot \mathbf{p}_f = \mathbf{p}_f \cdot \mathbf{p}_2 = 0$ ($\mathbf{p}_i \cdot \mathbf{p}_2 = -\mathbf{p}^2 \cos \phi$), we find the first nonzero contribution to the asymmetry factor

$$\delta = \frac{64(2\pi)^2 Z^4 \alpha^4 m^2 A^2}{(4E^2 - \mathbf{q}^2)} \mathbf{p}^4 \frac{\mathbf{q}^4}{Z^2\alpha^2(4E^2 - \mathbf{q}^3)}$$
$$= Z^2\alpha^2(\ln 2)^2 \frac{\beta^2(1 - \beta^2)}{(2 - \beta^2)^2}$$

in agreement with the answer quoted by Dalitz, (1951). It is of course only the first approximation, and for large-Z nuclei, higher order corrections will be far from negligible.

9. PARTIAL SPIN SUMS IN QED

Much of the simplicity and elegance of the above method comes from the fact that we were considering a single electron. As a more complicated example, consider electron–muon scattering. For each of the fermion lines, one has a scattering operator with an M of the form

$$M = eD_F \gamma^a J_a$$

where D_F is the photon propagator and J_a is given by

$$J_a = e\langle \tilde{u}_s \, \gamma_a \psi_2 \gamma_0 \rangle s$$

Here $\langle \ldots \rangle_S$ denotes the scalar and Σ projection, and provides the usual complex structure, u_s is the normalized final state of the other particle, and ψ_2 is the incoming state of the other particle. Now we can proceed to calculate cross sections as before if we sum over the final spin of the other particle to get the result

$$|T_{fi}|^2 = \frac{2e^4 \rho_1 \rho_2}{m_1 m_2 q^4} [p_1' \cdot p_2' p_1 \cdot p_2 + p_2' \cdot p_1 p_2 \cdot p_1' - m_1^2 p_2' \cdot p_2$$
$$+ m_2^2 p_1' \cdot p_1 + 2m_2^2 m_1^2 - [q \cdot (\hat{S}_1 \wedge p_1)] \cdot [q \cdot (\hat{S}_2 \wedge p_2)]]$$

We could also calculate final polarizations and spins in the same way as before.

While the scattering operator approach offers little advantage if one is just interested in unpolarized cross sections, it may still be useful for calculating spin-dependent results. If we are just interested in the spin dependence of a particular fermion line, the scattering operator approach works well once the spins of the other particles have been summed over. So in the scattering operator approach, we still have to perform a spin sum, but only over the

spins of the other particles. For example, above we summed over the final spin of the second fermion. We could of course introduce spin projection operators to single out particular spins of the other particles in the usual way.

10. CONCLUSION

We have shown how Hestenes' STA formulation of Dirac theory can be used to provide an elegant method of performing cross-section calculations. The logic of calculating cross sections is simplified considerably and spin is handled straightforwardly. There are no unnecessary spin sums and spin dependence is manifest in the spin bivector dependence of the scattering operator. It is a simple matter to calculate spin precessions, polarizations, and spin-dependent results, and the results obtained are expressed in terms of physical spin bivectors and the other scattering parameters. To perform unpolarized calculations, we simply average over spins.

The multiparticle case is rather more complicated. We do not yet have a neat method for performing arbitrary spin-dependent calculations, and have to involve spin sums over terms involving complex conserved currents. However, we can still write down a scattering operator for any given fermion line and retain the benefits of the scattering operator approach for calculations involving the spin of the particle. For clarity, we have only considered electron scattering, but all our results are easily extended to positron scattering and electron–positron annihilation (Lewis, 2000).

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